Interaction distance: patterns in entanglement

Christopher J. Turner, Konstantinos Meichanetzidis,
Zlatko Papic, Jiannis K. Pachos

School of Physics and Astronomy, University of Leeds

6th November 2017
Verona, QTML 2017

Nat. Commun. 8. 14926 (2017)
arXiv:1705.09983

Motivation

Many-body physics is hard...

- How distinct are the ground states of interacting systems of fermions from non-interacting systems?
- How good are non-interacting and mean field approximations to interacting physics?
- Can new perspectives be drawn from *quantum information* theory?
- Can we do all this more efficiently using some ideas from *machine learning*?

Outline

Free fermions and interaction distance

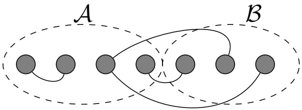
Example: Ising model in a magnetic field

Interaction distance and supervised learning

Conclusions

Entanglement spectrum

We partition our system and its Hilbert space $\mathcal{H}$ into two subsystems $\mathcal{A}$ and it's complement $\mathcal{B}$.



The reduced density matrix for the pure state $|\psi\rangle$ in subsystem $\mathcal{A}$ is the partial trace

$$\rho_{\mathcal{A}} = \mathrm{tr}_{\mathcal{B}} |\psi\rangle\langle\psi| \tag{1}$$

and the corresponding entanglement Hamiltonian

$$H_E = -\ln \rho_{\mathcal{A}} \tag{2}$$

has eigenvalues ξ_k, known as the *entanglement spectrum*[1]. What information can be found in the entanglement spectrum?

[1] Li and Haldane 2008.

Entanglement spectrum of non-interacting fermions

The entanglement spectrum f for an eigenstate of a system of *free fermions* is built from a set $\{\varepsilon\}$ of single particle entanglement energies[2] by

$$f(\sigma) = \mathrm{eig}(-\log \sigma) = \left\{ z + \sum_r n_r \varepsilon_r \right\}_{n_r=0,1} \quad \forall \sigma \in \mathcal{F}$$

This structure is intuitively similar to the many-body energy spectrum where the spectrum is built out of populating *independent* modes.

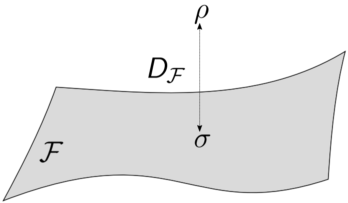
[2] Peschel 2003.

Interaction distance

In order to quantify the dissimilarity of an interacting system to the class of free fermion systems we introduce the *interaction distance*[3]

$$D_{\mathcal{F}}(\rho) = \min_{\sigma \in \mathcal{F}} D(\rho, \sigma)$$

where $D(\rho, \sigma) = \frac{1}{2}\mathrm{tr}\{\sqrt{(\rho - \sigma)^2}\}$ is the trace distance.



[3] Turner et al. 2017.

Properties of $D_{\mathcal{F}}$

It has an operational interpretation as measuring the distinguishability of the state from an eigenstate of a non-interacting Hamiltonian with an optimal measurement local to the reduced system[4].

$$D(\rho, \sigma) = \max_{P} \operatorname{tr} P(\rho - \sigma) \tag{3}$$

In density functional theory (DFT) a free description is found which reproduces the expectation values of functions of density operators, $D_{\mathcal{F}}$ bounds the accuracy for other observables [Patrick et al. incoming preprint].

[4] Englert 1996.

Unitary orbits

The manifold $\mathcal{F}$ contains all unitary orbits because each sigma is unitarily diagonalisable

$$\sigma = \exp\{z + \sum_r \varepsilon_r c_r^\dagger c_r\} \tag{4}$$

effecting a transformation $c_r \mapsto U c_r U^\dagger$ which preserves the CAR algebra.

Notice however that the trace distance is minimised within a unitary orbit when σ and ρ are simultaneously diagonal and in rank-order[5].

This simplifies $D_{\mathcal{F}}$ to depend only on the spectrum[6]

$$D_{\mathcal{F}}(\{\xi\}) = \min_{\{m\}} \frac{1}{2} \sum_k \left| \mathrm{e}^{-\xi_k} - \mathrm{e}^{-f_k(m)} \right|$$

[5] Markham et al. 2008.

[6] Turner et al. 2017.

Ising model

$$H_{\pm} = -\sum_{j=1}^{L} (\underbrace{\pm\sigma_j^x \sigma_{j+1}^x + h_z \sigma_j^z}_{\text{free}} + \underbrace{h_x \sigma_j^x}_{\text{interaction}}) \quad (5)$$

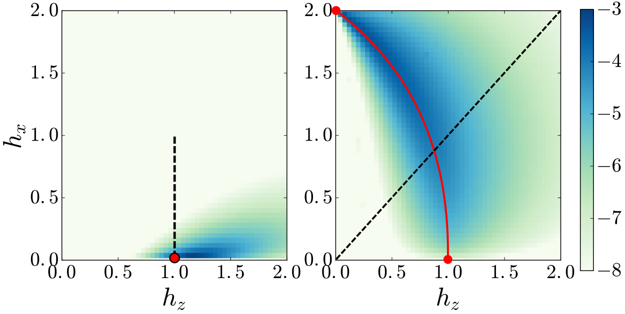


Figure: $D_{\mathcal{F}}$ for the ferromagnetic (left) and antiferromagnetic (right) Ising model. $L = 16$ and periodic boundary conditions.[8]

[7] Turner et al. 2017.

$D_{\mathcal{F}}$ as an inverse problem

Free fermion structure is characterised by a function

$$\text{expand} : \mathbb{R}_{>}^{N} \to \mathbb{R}_{>}^{2^N} \tag{7}$$

between spectra (multisets).

A method of solution for the problem of finding $D_{\mathcal{F}}$ and σ is a weak inverse form expand, which minimises $D_{\mathcal{F}}$ for input outside the image of expand.

$$\text{expand} \circ \text{factor} \circ \text{expand} = \text{expand} \tag{8}$$

$$\text{factor} \circ \text{expand} \circ \text{factor} = \text{factor} \tag{9}$$

$$\mathbb{R}_{>}^{2^N} \xleftarrow{\text{expand}} \mathbb{R}_{>}^{N} \xleftarrow{\text{factor}} \mathbb{R}_{>}^{2^N} \xleftarrow{\text{expand}} \mathbb{R}_{>}^{N} = \mathbb{R}_{>}^{2^N} \xleftarrow{\text{expand}} \mathbb{R}_{>}^{N} \tag{10}$$

A linear approximation

If we ignore the distinction between vectors and multisets then expand becomes a linear map E

$$\mathrm{expand} \sim E : \mathbb{R}^N \to \mathbb{R}^{2^N}. \tag{11}$$

As a matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{12}$$

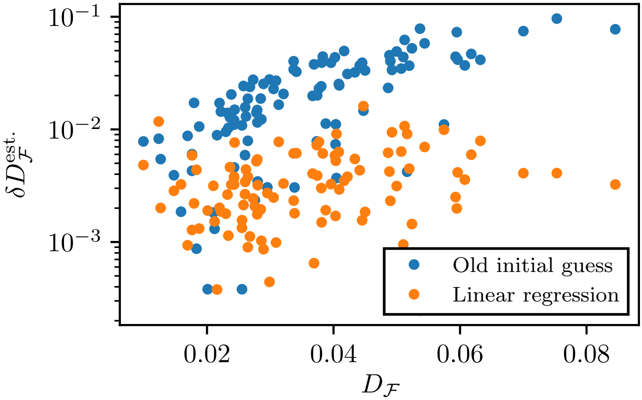
containing all bitstrings as rows.

It has linear weak inverses (i.e. Moore-Penrose pseudoinverse).

Results from linear regression

Least squares δ^2 solution for the linear system

$$\varepsilon = F\xi + \delta \tag{13}$$



Future directions

- Least-squares cost function is not appropriate, it favours getting high energy structure right although it's Boltzmann factor is negligible.
- A linear model can't capture the ordering structure – this will also be replaced by something more sophisticated.
- Could this be done with unsupervised learning?