Ergodicity breaking: fragmentation and quantum scars

C.J.Turner

14th April 2025, TU Delft

Best regards to D. A. Abanin, A. A. Michailidis, Z. Papić, M. Serbyn, J.-Y. Desaules, K. Bull, M. Kovács, L. Masanes, D. Paszko, D. Rose, A. Pal, M. Szyniszewski, B. Mukherjee, R. Melendrez, H. Changlani

Outline

Weak ergodicity breaking

Fragmentation

Scars in the PXP model

Alternting Heisenberg model






Forms of ergodicity breaking

What avenues are there for a quantum system to avoid equilibrium?

- Non-interacting,
- integrable,
- many-body localised systems

These generically fail to equilibrate. Ergodicity is completely broken.

Q: What would be weak ergodicity breaking?

- Sensitivity to initial conditions
- Some atypical eigenstates

We have two answers to this question now,

- Fragmentation
- Quantum scars

Hilbert-space Fragmentation

It can happen that the connectivity of states induces by H forms multiple connected components. Representation theory framework (see Moudgalya *et al.* PRX 2022), for a Hamiltonian, $H = \sum_i h_i$, define a pair of C^*-algebras,

$$\mathcal{A} = \langle h_i \rangle \qquad \mathcal{C} = \mathcal{A}' = \{O : [O, \mathcal{A}] = 0\} \tag{1}$$

The state space decomposes into a direct sum, c.f. Schur-Weyl duality,

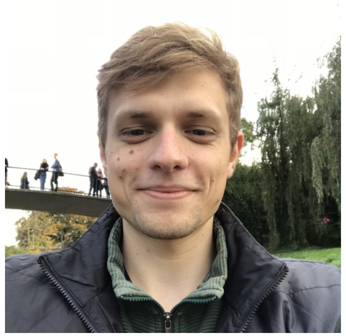
$$\mathcal{H} = \bigoplus_{\lambda} \left(V_{\lambda}^{\mathcal{A}} \otimes V_{\lambda}^{\mathcal{C}} \right) \tag{2}$$

The splitting of H is what makes this non-trivial.
We can also find this in other settings:

- Quantum circuits
- Open quantum systems

Fragmentation in open quantum systems

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = -i\,[H,\rho] + \sum_j \kappa_j \left(2F_j\rho F_j^\dagger - \{F_j^\dagger F_j, \rho\}\right)$$



D. Paszko

D. Rose

Hamiltonian terms $H = \sum_j Z_{j-1}X_jZ_{j+1}$ and jump operators $F_j = Z_{j-1}Z_{j+1}$.

These sums don't include any terms which would go over the open boundaries.

- Hamiltonian is an SPT phase.
- Zero modes are (approximate) strong symmetries before dissipation.
- Some of them then become *weak* symmetries after dissipation.
- Information is recoverable if you can make the jumps *observable*

This and more is all in arXiv:2310.09406 and not what this talk is about.

OSF: Superoperator algebras

Natural generalisation of bond $\mathcal{A}$ and commutant $\mathcal{C}$ algebras,

$$\mathcal{A} = \langle u_j = -i\,\mathrm{ad}_{H_j},\ d_j = \mathrm{Ad}_{F_j} \rangle$$
$$\mathcal{C} = \{O : [O, u_l] = 0, [O, d_l] = 0\}$$

From the double commutant theorem we get a representation theoretic structure like the Schur-Weyl duality,

$$\mathcal{H} = \bigoplus_{\lambda} V_{\mathcal{A}}^{(\lambda)} \otimes V_{\mathcal{C}}^{(\lambda)}$$

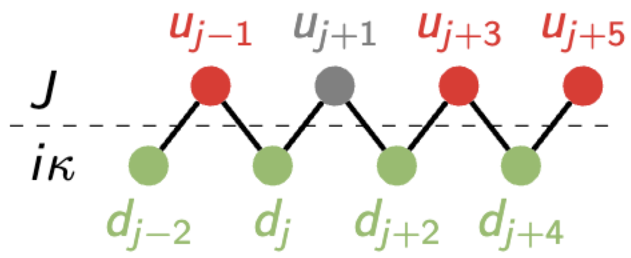
Operator dynamics of a Pauli M,

$$u_j M = -i\,\mathrm{ad}_{H_j} M = \begin{cases} 0 & \text{if } [H_j, M] = 0 \\ -2iH_j M & \text{otherwise} \end{cases} \qquad d_j M = \begin{cases} +M & \text{if } [F_j, M] \\ -M & \text{otherwise} \end{cases}$$

So $(u_j)^2 \in \mathcal{C}$

OSF: Frustration graphs

The Lindblad terms either commute or anti-commute so we can summarise $\mathcal{A}$ represented onto a $(u_j)^2$ fragment with a frustration graph,



and choose another presentation for this algebra with the same properties (see Chapman *et al.* Quantum 2020). It's the TFIM!

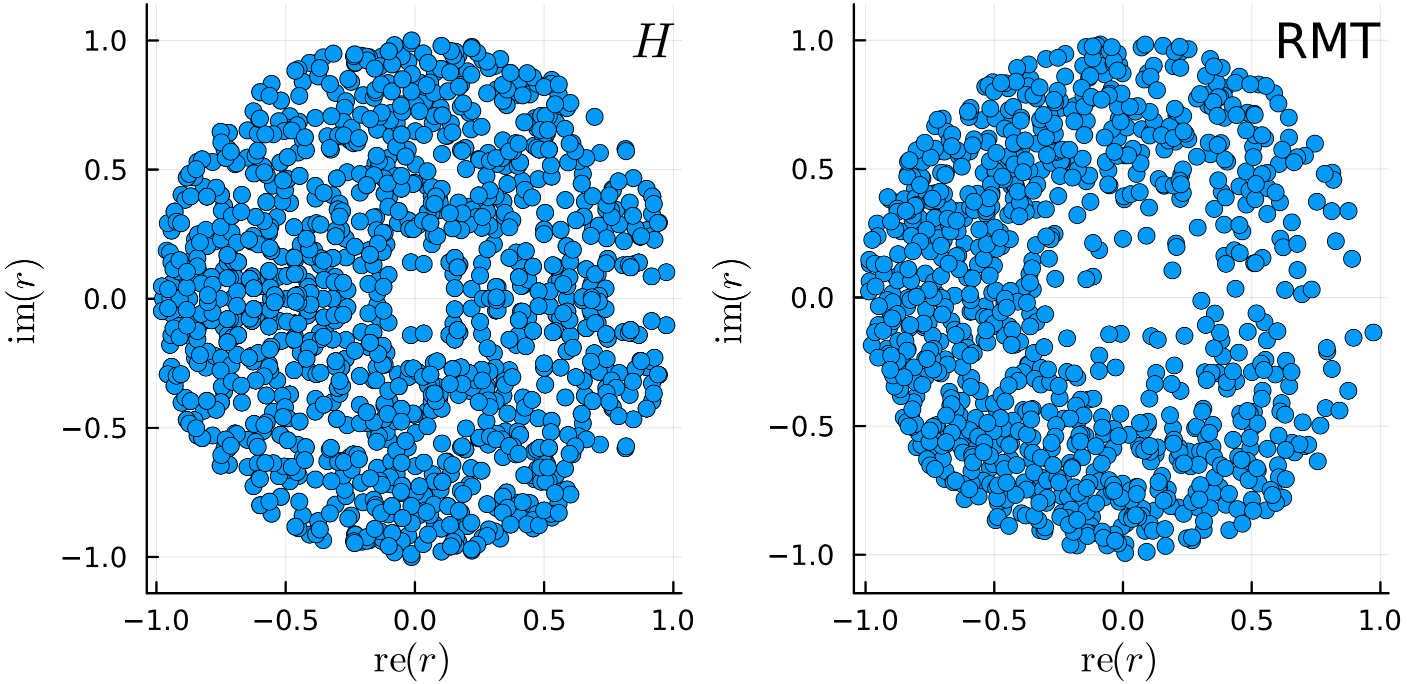
- If $(u_j)^2 = 0$ in a representation then that Ising term is missing, dividing the system into subsystem fragments.
- Fields can be missing at the boundary due to open boundary conditions.
- There's another copy of the ising chain for the other parity of terms.

OSF: Effective Model

- The effective model is a non-Hermitian transverse-field Ising model

$$H = \sum_j J\sigma_j^X \sigma_{j+1}^X + i\kappa \sum_j \sigma_j^Z$$

- Sometimes the boundary fields are missing again. This embeds the global zero modes from before.
- Complex level spacing ratio after resolving symmetries [Sá *et al.* PRX 2019]

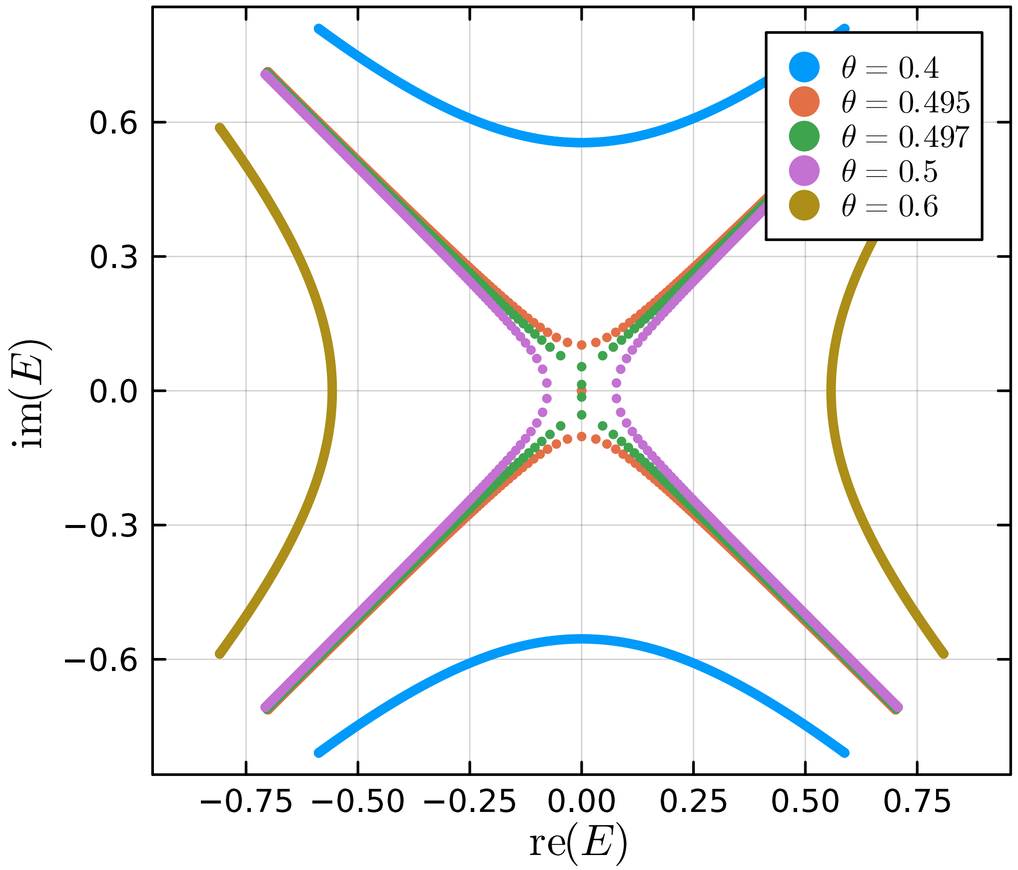


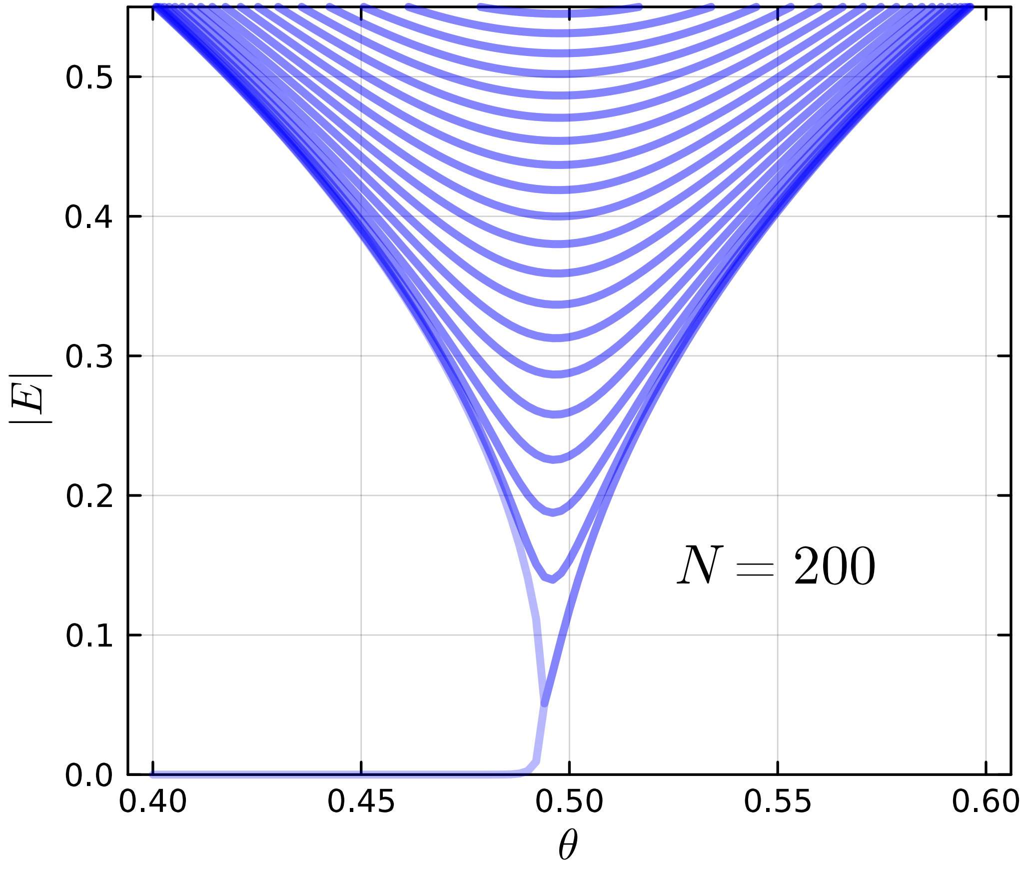
$$r_a = \frac{E_{\mathrm{nn}} - E_a}{E_{\mathrm{nnn}} - E_a}$$

OSF: Integrability and phase transition

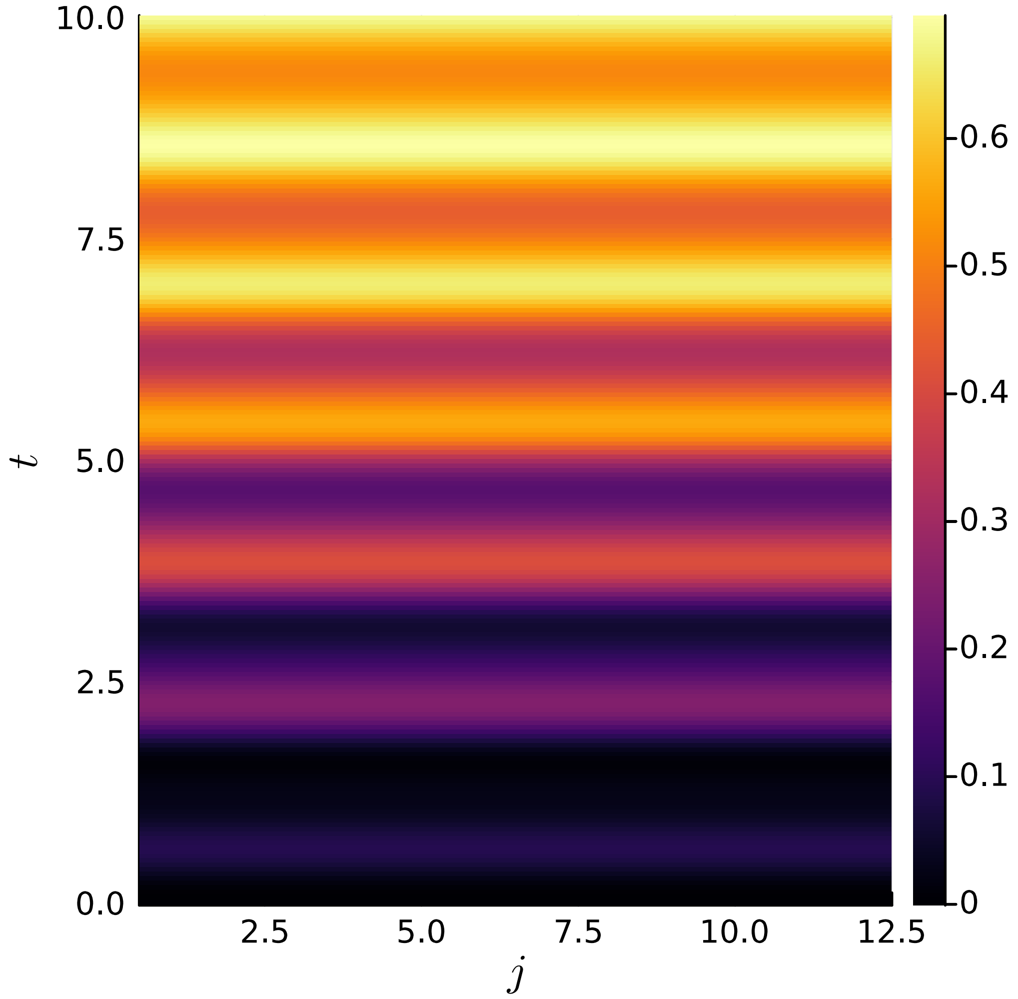
$$H = J \sum_j \gamma_{2j-1}\gamma_{2j} + i\kappa \sum_j \gamma_{2j}\gamma_{2j+1} = i\gamma^T A \gamma$$

$$J = \cos(\theta\pi/2),$$
$$K = \sin(\theta\pi/2)$$





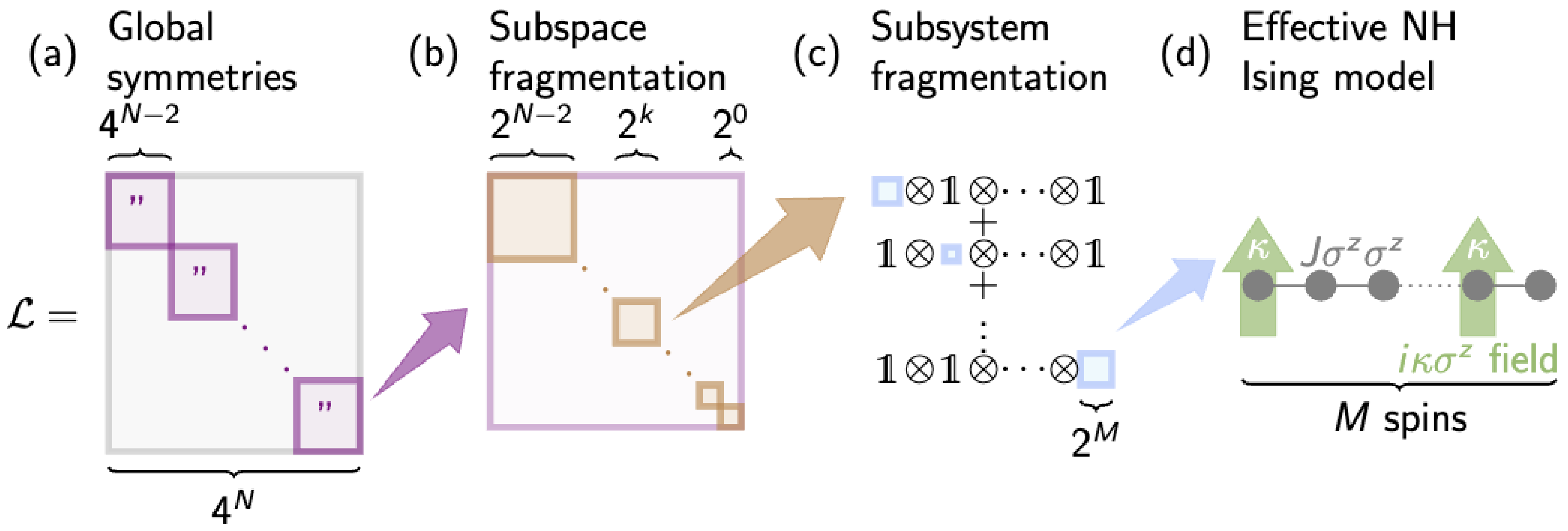
OSF: Dynamical consequences



- Quench from (rapid cooling) ground state $|\phi\rangle$ of κ-dominated phase to J-dominated phase.
- $|\phi\rangle$ is roughly an extremal Y eigenvector.
- Looking at observables $\langle\phi(t)|\,\sigma_j^Z\,|\phi(t)\rangle$.
- Dynamic phase transition, oscillations in the κ order parameter.
- Eventually dissipation will win and system equillibrates.

OSF: Summary

arXiv:25XX.XXXX soon (hopefully)

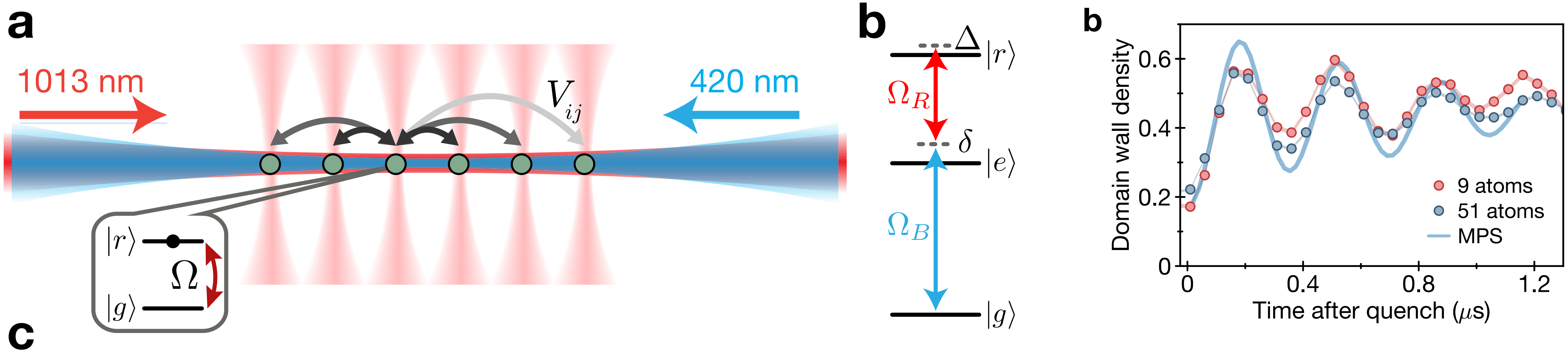


- Fragmentation can naturally be generalised to Lindblad master equation as operator-space fragmentation.
- We can observe a non-Hermitian dynamical phase transition in the operator dynamics

Rydberg atom system

Bernien et al. Nature 551(7682), 579 (2017)

- Cold neutral atoms resonantly driven with a laser to produce Rabi oscillations.



- If the interaction energy is very large the we get the Rydberg blockade where adjacent excited atoms ●● are forbidden.
- Long-lived oscillations were observed, but only from very specific states.

Effective PXP model

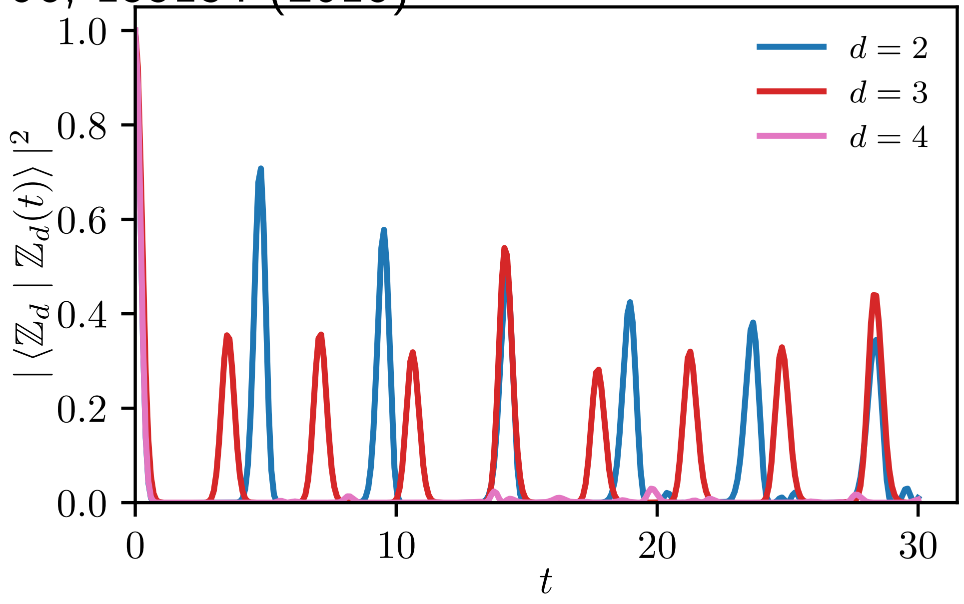
Turner et al. Nat. Phys. 14, 745 (2018) Turner et al. PRB 98, 155134 (2018)

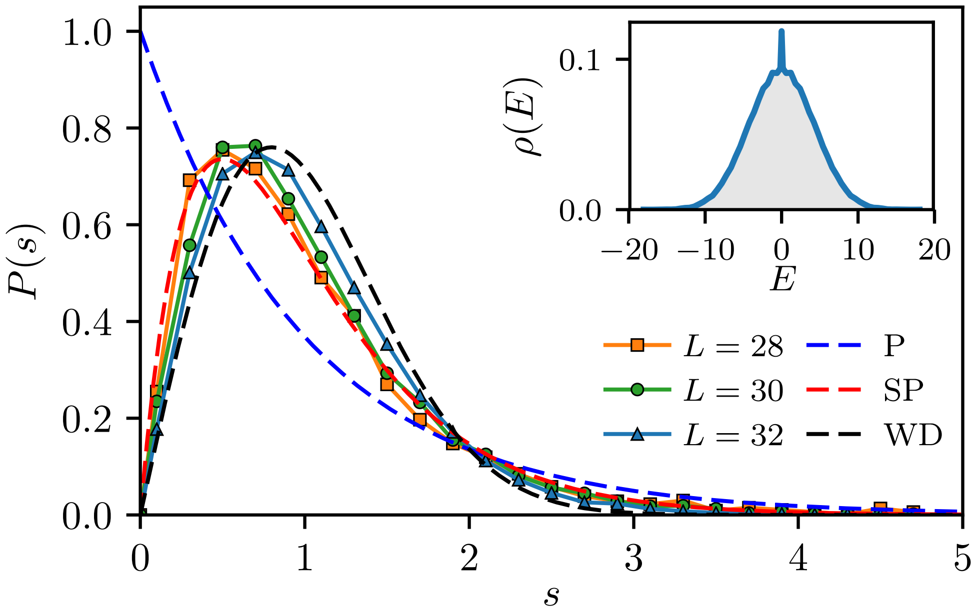
In the blockade regime, the effective Hamiltonian is,

$$H = \sum_{j=1}^{N} P_{j-1} X_j P_{j+1} \qquad \circ\circ\circ \leftrightarrow \circ\bullet\circ \qquad (3)$$

- Oscillations in local observables is caused by a periodic quantum revival.
- The level statistics is Wigner-Dyson, which rules out strong ergodicity breaking.
- There's an exponential degeneracy at zero energy.

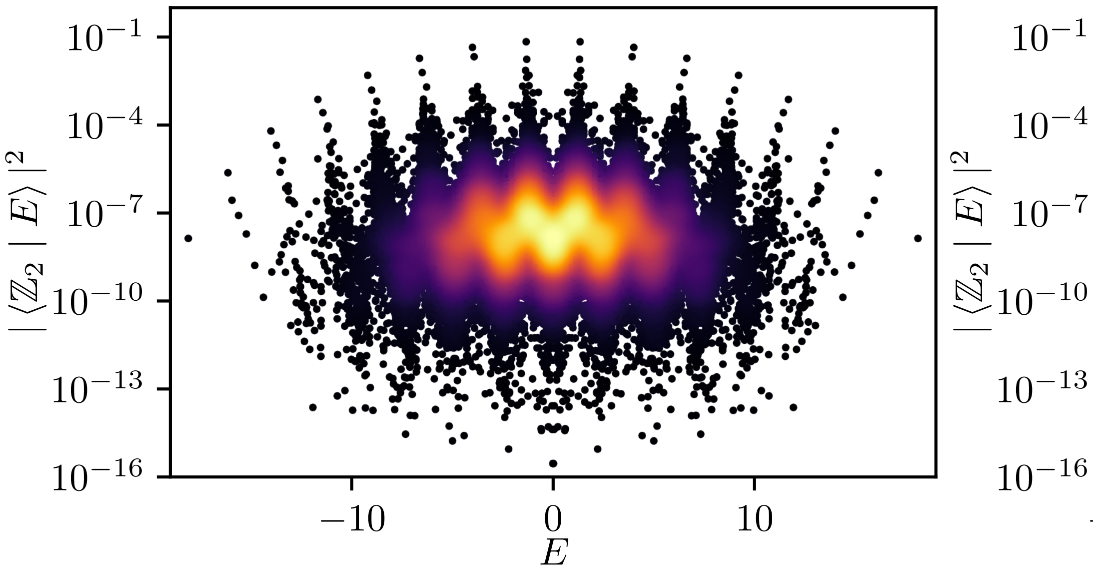
What's going on?





Scarred eigenstates in PXP

- A small number of special eigenstates with anomalous overlap with the Néel state.
- They have approxiamtely equally spaced eigenvalues, converging with N.
- This explains the oscillatory dynamics.
- Most of these eigenstates are not eigenstates of local frustration-free Hamiltonians.
- With a couple of exceptions, [Lin + Motrunich]



Recall the Shiraishi-Mori construction for embedding frustration-free ground states Z as scar states,

$$H = \sum_i h_i P_i + H' \qquad [H', P_i] = 0 \qquad [P_i, P_j] = 0 \qquad Z = \bigcap_i \ker P_i \tag{4}$$

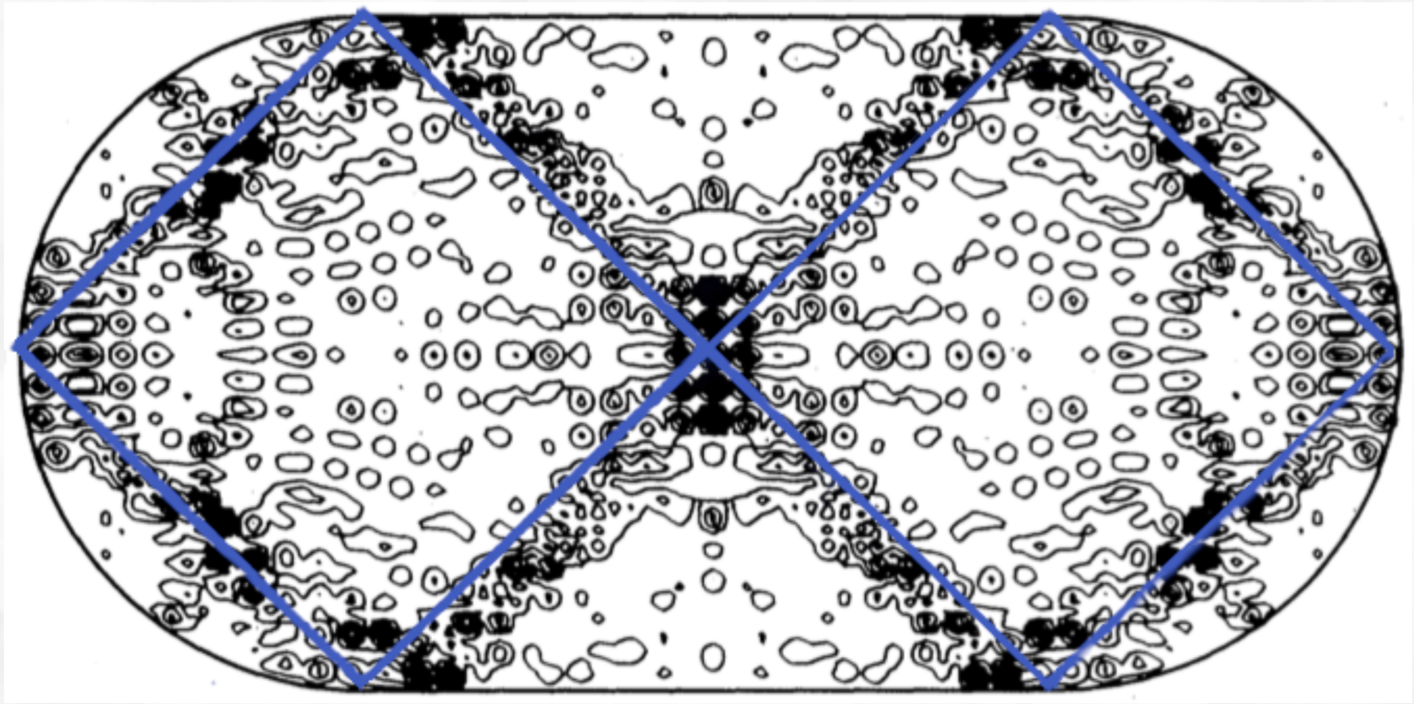
Most examples of many-body quantum scars in the literature can be put into this form.

But these are not one of them.

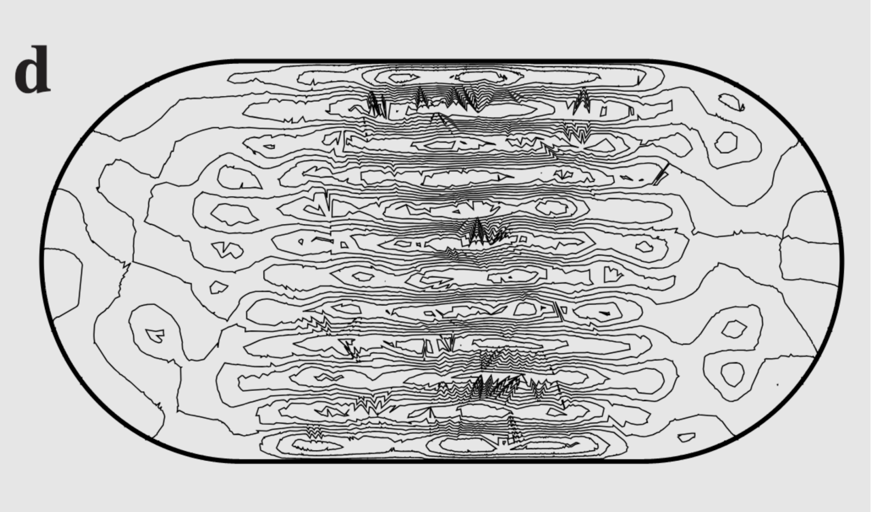
Quantised billiars

Heller PRL 53(16), 1515 (1984)

- Unstable periodic orbits of the chaotic classical billiard imprint upon a wavefunction after quantisation.
- This is surprising! One might expect that if a particle's position gets blurred out upon quantisation that it would diverge rapidly from unstable trajectories.



Quasi-modes and scarred eigenstates

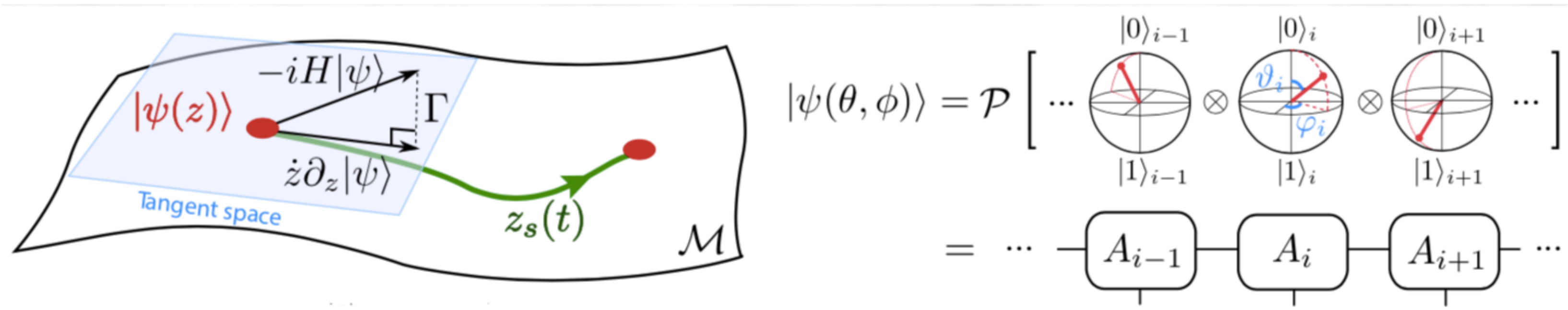


- Approximate eigenstates of the form $\phi = \chi sin(ny)$ for suitable χ [O'Conner and Heller 1988]. These are like "bouncing balls" trajectories? But how can these be connected to the exact eigenstates?
- If the quasi-mode has an energy variance K^2 and there's at most M eigenstates in a $4K$ interval around the quasi-mode, then there exists eigenstates with anomalously large overlap [Zelditch 2004]. This is the corresponding scarred eigenstate.
- Constructing the quasi-modes was simple enough, but showing that the density of states is non-pathological was *much* more challenging [Hassell 2010].

- The flavour of these quantum scars is quite different to those following Shiraishi-Mori. Can we follow this recipe in a many-body system?
- We could call this distinction *algebraic* vs *analytic* quantum scars.

Coherent states and variational dynamics

- The most classical states for many systems are the coherent states. There's no suitable frame of traditional coherent states here because of the constraint.



- We can take the next best thing: the quantum dynamics is restricted into a class of states $\mathcal{M}$ using the variational principle (TDVP). See [Ho et al. PRL 2019] and [Michailidis et al. PRX 2020].
- We can view this as a mere variational approximation, but we can also view it as a classical system in its own right.

Path integral quantisation and correspondence principle

Recall the Feynman path integral construction for a particle in quantum mechanics. We take a propagator, subdivide it and insert resolutions of the identity.

$$\mathbb{1} = \int \mathrm{d}x \, |x\rangle \langle x| \tag{5}$$

Keep refining this process and you have a path integral.
For some provided measure μ, we can construct a frame operator,

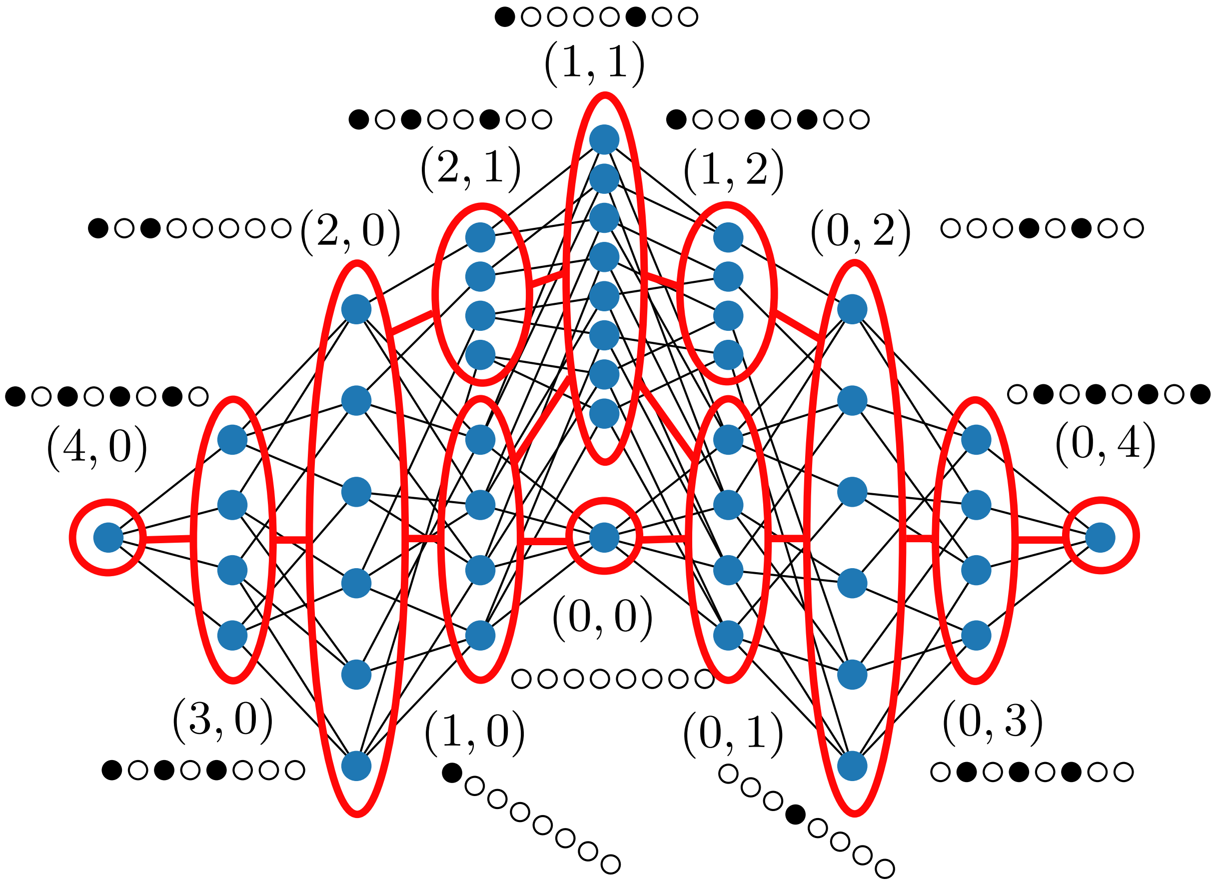
$$S_\mu = \int \mathrm{d}\mu(x) \, |\psi(x)\rangle \langle \psi(x)| \tag{6}$$

Can we choose μ such that S_μ is a resolution of the indentity on some subspace? Yes! From $\mathcal{M}$, you can get the identity on a vector space $\mathcal{K}$.

Quasi-modes and permutation symmetry

Turner *et al.* PRX 11, 021021 (2021)

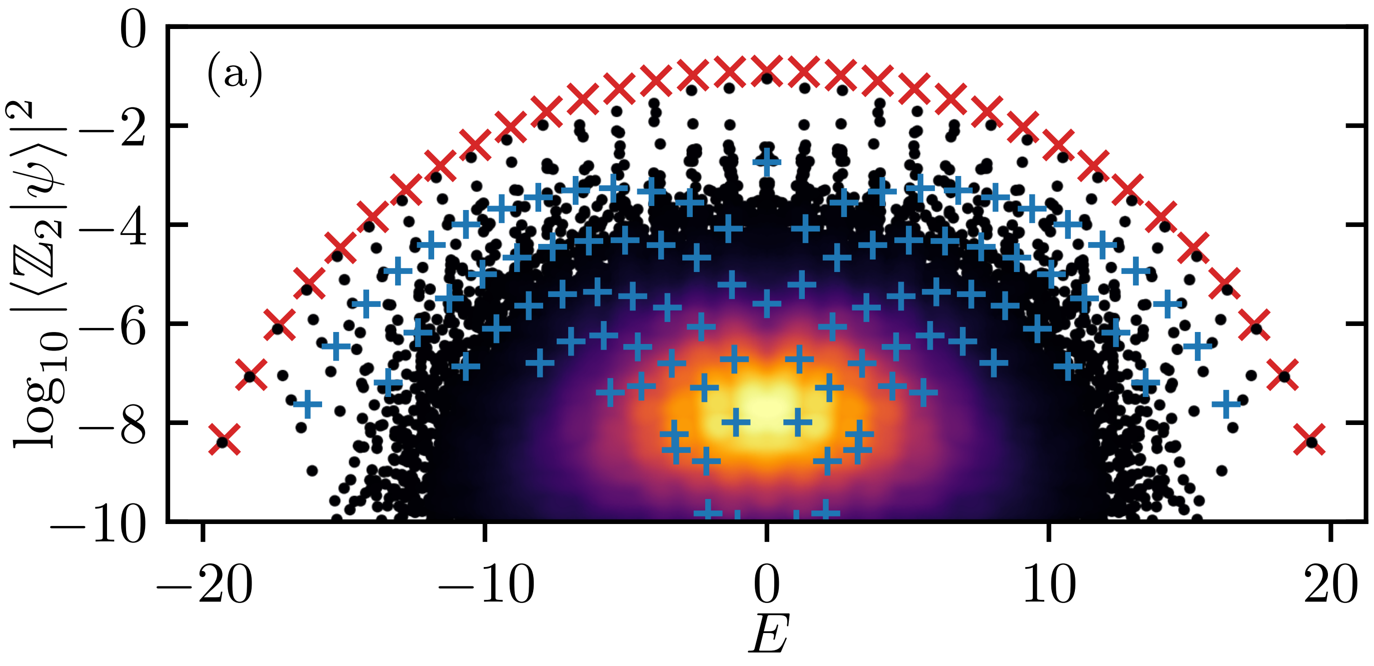
- $\mathcal{K}$ is a space generated by states with fixed numbers of excitation on each sublattice. It's a sublattice respecting but otherwise permutation-invariant subspace projected into the constrained Hilbert space.
- It has $O(N^2)$ dimension and matrix-elements can be found with some combinatorics.



Quasi-modes and permutation symmetry

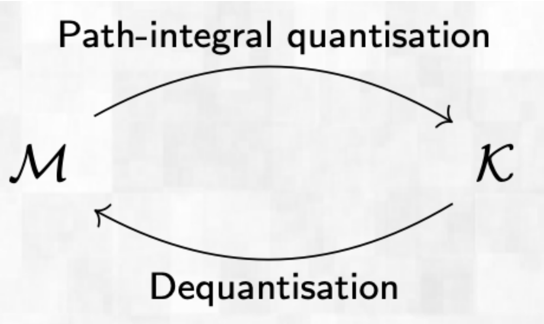
Turner *et al.* PRX 11, 021021 (2021)

- $\mathcal{K}$ is a space generated by states with fixed numbers of excitation on each sublattice. It's a sublattice respecting but otherwise permutation-invariant subspace projected into the constrained Hilbert space.
- It has $O(N^2)$ dimension and matrix-elements can be found with some combinatorics.
- We can construct quasi-modes in $\mathcal{K}$.

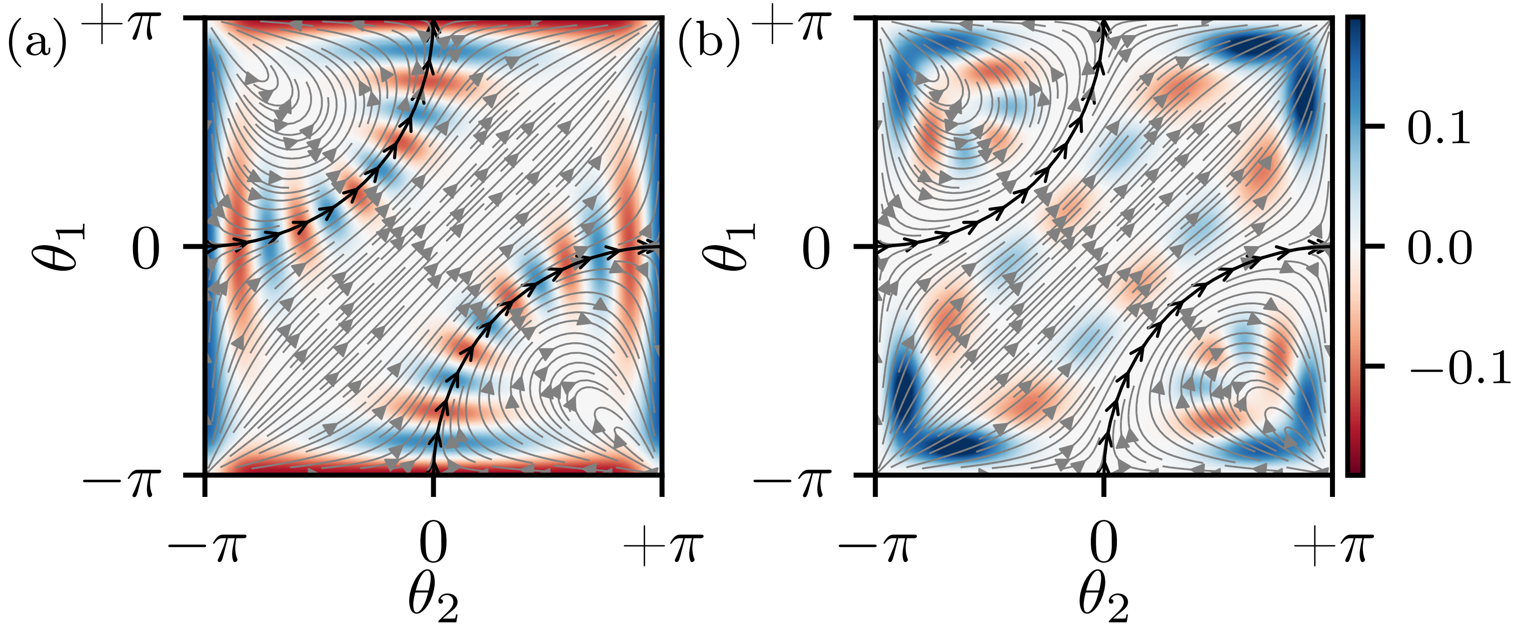


Scarring for Rydberg atoms

Turner *et al.* PRX 11, 021021 (2021)



Gives a picture of the quasi-modes in $\mathcal{K}$ as wavefunctions on $\mathcal{M}$.



The good quasi-modes take the appealing form of standing waves along the classical periodic orbits.

Summary

A strange many-body system which displays a new kind of ergodicity breaking.
An analogy:–

Many-body quantum scar		Single-particle quantum scar
TDVP regular trajectories	$\leftrightarrow$	Classical periodic orbits
Conditional quantum revival	$\leftrightarrow$	Oscillatory wavepackets
ETH-violating atypical eigenstates	$\leftrightarrow$	Scarred wavefunctions (non-uniform measure)
Our new quasi-modes	$\leftrightarrow$	Boucing-ball quasimodes

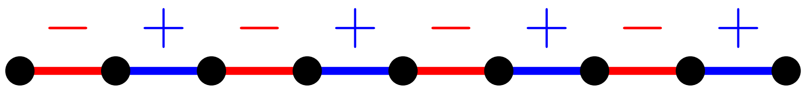
- New-and-improved quasimodes much easier to work with than FSA.
- With connections to a classical limit.
- The subspce $\mathcal{K}$ represents mean-field or coherent-state physics.
- Approach becomes exact in an $S \to \infty$ limit. See recent work by Markus Müller.

Alternating Heisenberg chain

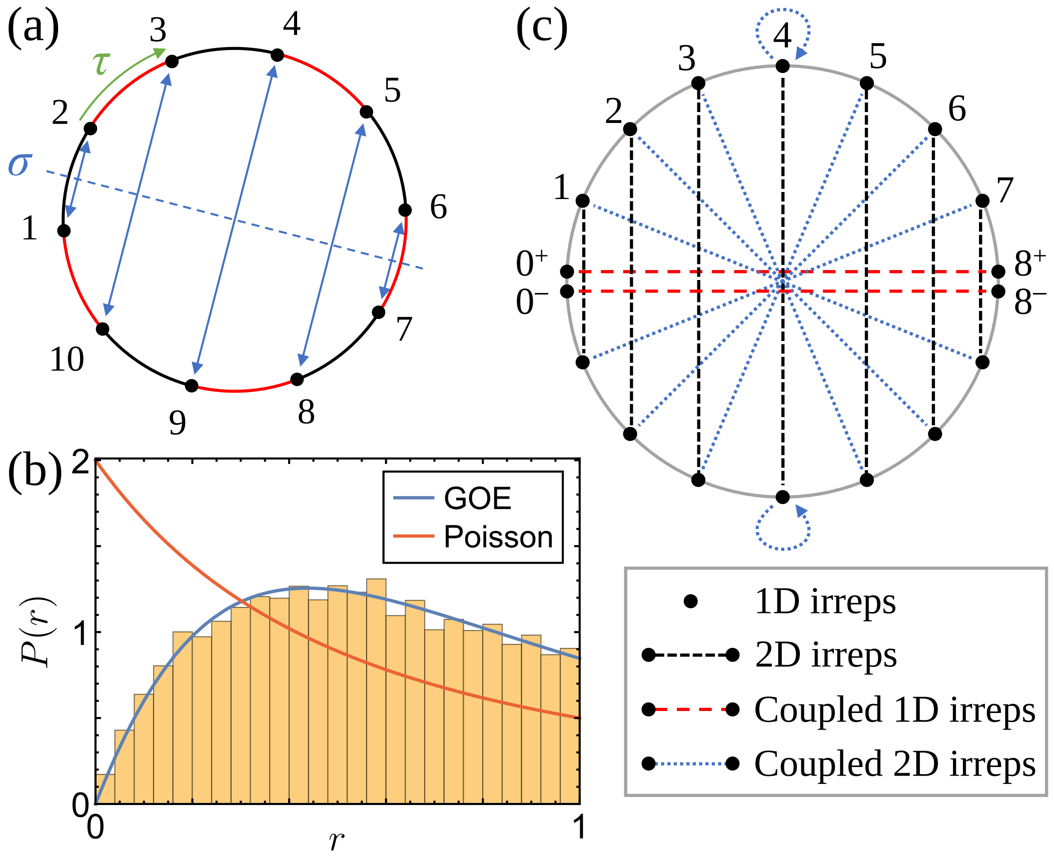
C. J. Turner, M. Szyniszewski, B. Mukherjee, R. Melendrez, H. J. Changlani, A. Pal — arXiv:2407.11956

Spin-$\frac{1}{2}$ Heisenberg chain except the coupling alternates in sign,

$$H = \sum_{j=1}^{2N} (-)^j S_j \cdot S_{j+1} \qquad (7)$$



- Exponential degeneracy at zero-energy – in the middle of the spectrum. This is due to antisymmetry.
- We find several different kinds of scar states in here.



Bethe-ansatz scars

arXiv:2501.14017 R. Melendrez, B. Mukherjee, M. Szyniszewski, C. J. Turner, A. Pal, H. J. Changlani

- Bethe Ansatz states are states of particles with well defined individual momentum,

$$|k_1, \cdots, k_m\rangle \propto \sum_{x_1<\ldots<x_m} e^{i\sum_a x_a k_a} |x_1, x_2, \cdots, x_m\rangle \quad (8)$$

$$|\psi\rangle = \sum_{\pi\in S_m} \alpha_{\pi(k_1,k_2)} |k_1, k_2\rangle \quad (9)$$

- We generalise this a small amount taking superpositions of small numbers of particle momentum sets.
- For magnon numbers $m = 2, 3$ we find all the zero-energy states.
- For $m = 4$, we find many solutions numerically (but not all the states).
- We have Bethe ansatz-like states as exact eigenstates in the interior of the spectrum.
- Different to the asymptotic Bethe ansatz.

H. J. Changlani

R. Melendrez

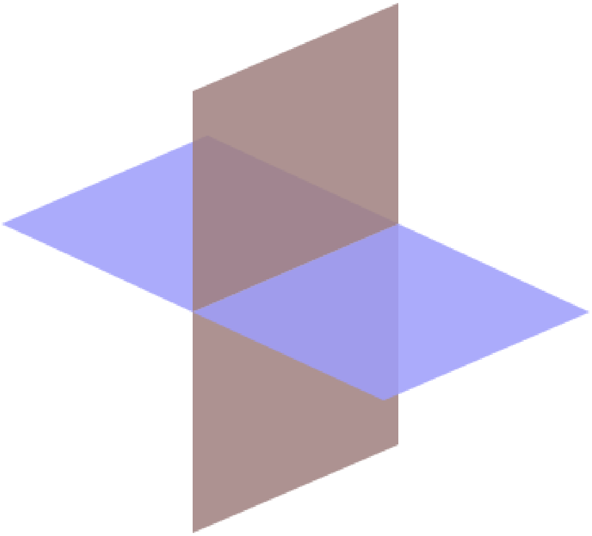
Bethe-Ansatz scars: details of solutions

- We add a sublattice degree of freedom, this is the same as combining each particle momentum k with $k + \pi$.
- For two magnons $m = 2$ and even N, the solutions have $k_1 = -k_2$.
- When N is odd you also get fractionalised momenta $(k + \pi/2, k + \pi/2)$.
- For $m = 3$, we take $(0, 0, 0)$, $(k, -k, 0)$. Again additional fractionalised solutions for odd N.
- For $m = 4$, we take $(0, 0, 0, 0)$, $(k_1, -k_1, k_2, -k_2)$, $(k_1, -k_1, 0, 0)$, $(k_2, -k_2, 0, 0)$, $(k_1, -k_1, k_1, -k_1)$ and $(k_2, -k_2, k_2, -k_2)$.

We're using a fairly flexible Ansatz so we must be careful: is finding solutions significant?

- Dimension of Ansatz is $O(N^2)$.
- Dimension of nullspace is $O(N^2)$.
- Dimension of enclosing symmetry sector is $O(N^3)$.

So a generic intersection would be zero-dimensional, but we find a growing number with N.



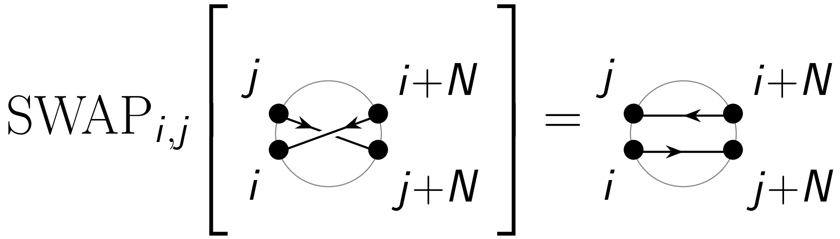
Symmetric tensor scars

arXiv:2501.14024 B. Mukherjee, C. J. Turner, M. Szyniszewski, A. Pal

Recently there has been interest in volume-law scarred eigenstates.
Provided N is odd, we have root states, pairing antipodal sites as zero-energy eigenstates,

$$|\Psi(v)\rangle = v^{\otimes N} \tag{10}$$

where v is either the singlet state or is any triplet state.



(11)

As all these states are degenerate, we can take linear combinations,

$$\mathrm{span}\,|\Psi(V_{S=0,1})\rangle \cong V^{Sym}_{S=0,1} \text{ as vector spaces} \tag{12}$$

allowing for state much more interesting than the root states.

B. Mukherjee

M. Szyniszewski

Symmetric tensor scars: Bell basis

From an (orthonormal) basis of $S = 1$, we can build an (orthonormal) basis of symmetric tensor states,

$$\left|\Psi_{n_1,n_2,n_3}\right\rangle \propto \sum_{\pi \in S_N} \pi\left(\left|T_1\right\rangle^{\otimes n_1} \otimes \left|T_2\right\rangle^{\otimes n_2} \otimes \left|T_3\right\rangle^{\otimes n_3}\right) \tag{13}$$

One choice is the Bell basis,

$$\left|T_X^B\right\rangle = \frac{1}{\sqrt{2}}|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \cong \frac{1}{\sqrt{2}}\sigma_X \tag{14}$$

$$\left|T_Y^B\right\rangle = \frac{1}{\sqrt{2}}|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle \cong \frac{1}{\sqrt{2}}\sigma_Y \tag{15}$$

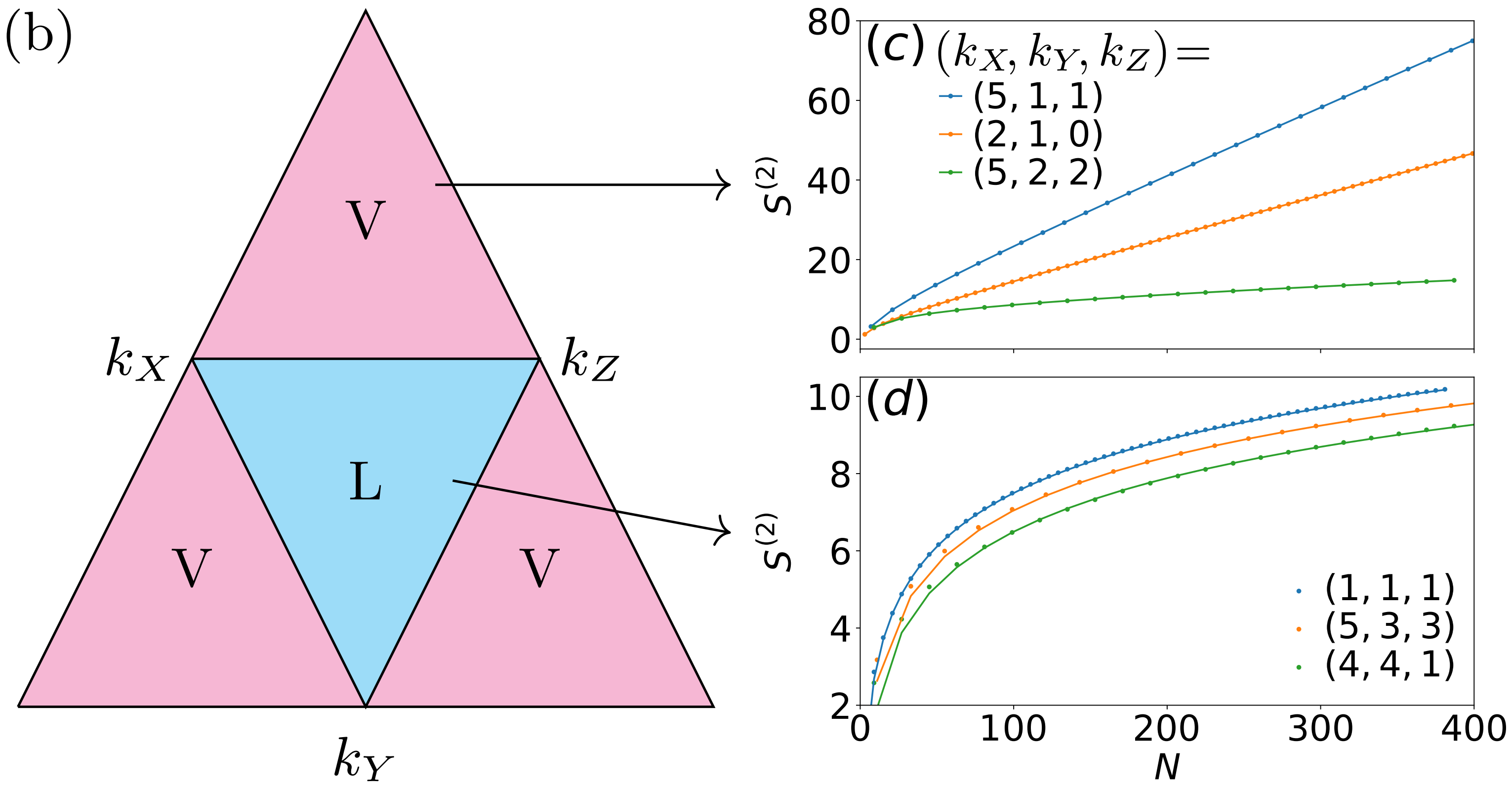
$$\left|T_Z^B\right\rangle = \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \cong \frac{1}{\sqrt{2}}\sigma_Z \tag{16}$$

Using the algebraic structure of the problem, we are able to compute the Renyi entropy efficiently and we can also extract asymptotics expansions in N.

$$S = -\log \operatorname{tr} \rho^2 = \Psi\Psi^\dagger \tag{17}$$

Entanglement in the Bell basis

Using $k_X = n_X/N$ etc.



When is a state thermal?

- Sometimes there are curious claims that the volume-law scar states are exactly-constructible thermal states.
- But what does it mean for something to be a thermal state?

We have a class of reference thermal states which are Gibbs states or maybe generalised Gibbs states. If a state is effectively indistinguishable from a reference state then it may as well be thermal. If we look at connected correlation functions on our states,

$$\langle S_i \cdot S_j \rangle = \begin{cases} 1/4 & \text{if } i \text{ and } j \text{ separated by } N \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

These states can't be distinguished from the infinite temperature / maximally mixed state by local expectation values.
So are these thermal states?

- A competent experimentalist is not an ant living inside their experiment.
- They can simultaneously place a probe on each side of the system and calculate this correlation function.
- This is like LO (local operation) vs LOCC (local operations and classical communication).

Alternating Heisenberg chain: summary

- Integrability-like scar states with $m = 2, 3, 4$ particles, featuring paired particle momenta.
- Alternative presentation of $m = 2$ states with high robustness to perturbation (arXiv:2407.11956).
- Many-body scar states with area-law, log-law or volume-law entanglement.
- These are scar states – not thermal states – because they're LOCC distinguishable.